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DIAGRAMMING LINEAR ALGEBRA: SOME RESULTS AND TRIVIAL PROOFS

by

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DIAGRAMMING LINEAR ALGEBRA: SOME RESULTS AND TRIVIAL PROOFS

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ABSTRACT

This paper explores the application of diagrammatic techniques to various problems in mathematics. Specifically, we will use special diagrams, often known as *spin networks* or *birdtracks*, to explore topics in linear and abstract algebra. We offer first a diagrammatic proof of Dodgson's Condensation Method for calculating the determinant, a curious result of a classical theorem by Jacobi. We then show the doodles' use in the Fricke-Vogt theorem for $\mathrm{SL}(3,\mathbb{C})$, and we offer a method for expressing any 3-trace diagram as a trace polynomial in the matrices, their inverses, and their transposes.

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I thank my advisor, Dr. Elisha Peterson, for introducing me to this topic and being patient with me until the concepts which fill the expository parts of this paper finally *clicked*. This thesis wouldn't have been possible were it not for his continued support and tutoring. I feel part of my undergraduate understanding of things is now inextricably tied to these doodles, and I hope my graduate experience will include them in some way too.

I thank my two faculty independent readers, Dr. Amanda Beecher and Dr. Jenny Fuselier, for their detailed comments. By taking the time to read and understand my thesis page by page, they were able to offer me far more extensive, helpful comments than I ever expected. I am very grateful for their time and effort.

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EXECUTIVE SUMMARY

This thesis focuses on two problems, one in linear algebra and one in abstract algebra, attacking both using certain diagrammatic techniques. The diagrams we use in this paper are sometimes called *spin networks*, having begun as a tool for calculations in quantum physics, but we have modified them to address more mathematical questions. Specifically, each diagram or doodle may be thought of as a function or mapping, and we may compute its value by evaluating smaller, well-defined pieces of the doodle. Starting with some very basic maps, we can build diagrams that elegantly represent ideas like the determinant, the trace, and more complicated things like a minor or the adjugate.

The first application of the doodles in this thesis concerns certain classical identities in linear algebra. Specifically, we introduce a curious method for calculating the determinant by *Charles Dodgson* (aka Lewis Carroll), and we demonstrate that it is actually a special case of a much more powerful theorem due to Jacobi. After introducing basic diagrammatic methods, we show that it is possible to prove this powerful theorem in a very simple, almost trivial way using diagrams.

The second area of study in this thesis deals with the application of diagrams to abstract algebra and, more specifically, *representation theory*. In this area of mathematics, homomorphisms from an abstract group to a matrix group allows us to apply the well-known properties of matrices towards understanding an abstract group. We may apply diagrammatic methods to help determine a sufficient set of trace relations and generators for particular matrix groups.

This second area of study is the focus of the thesis, and we offer two results. Concerning ourselves primarily with the matrix group $\mathrm{SL}(3,\mathbb{C})$, and calling the resulting diagrams *3-trace diagrams*, we show first a family of braid-like diagrams that seem to provide a prototype for all trace relations. We then present an *algorithm* for evaluating these 3-trace diagrams, which demonstrates that any 3-trace diagram represents a trace polynomial and shows how to find that polynomial. Finally, we use this algorithm to evaluate a braid-like diagram, giving our partial result.

Part of the goal of this thesis is to show the power these diagrams have in varied areas of mathematics. By showing results from linear and abstract algebra, we hope we have not only shed new light on these problems, but shown the power and elegance of the diagrams' as well.

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I. INTRODUCTION

Arguably the most powerful tool of problem-solving, in any discipline, is being able to draw or visualize the problem. In mathematics especially, doodles and diagrams often turn complex problems into simple, understandable questions and lead to elegant solutions. For example, think of the way graph theory turns puzzles like the Traveling Salesman and Four-Color Problems into visual riddles even someone totally unfamiliar with mathematics can immediately understand.

In this paper, we explore techniques for diagramming problems in linear algebra. With a few basic definitions, one can construct a system of diagrams which become very powerful tools for understanding the nature and properties of matrices. In particular, we first examine Charles Dodgson's *Condensation Method* for calculating the determinant, a related theorem of Jacobi's, and these theorems' surprising applications to other areas of mathematics. We also begin to explore the profound usefulness the diagrams have in dealing with trace algebras, and begin a proof of a generalization of a classical theorem by Fricke-Vogt to higher dimensions that demonstrates the elegance diagrams can lend to otherwise messy abstract computations. Central to this discussion will be our method for reducing any diagram to a trace polynomial.

A. SPIN NETWORKS, BIRDTRACKS, AND KNOTS

The types of diagrams we use in this paper have a lineage primarily propagated by physicists interested in simplifying and beautifying the difficult calculations and tensor calculus common in quantum theory. Although our diagrams will take a carefully different shape than the physicists', someone already familiar with theirs will feel at ease adjusting to the diagrams here.

In 1971, Roger Penrose pioneered a type of diagrams he called *spin networks* as an alternative method for modeling space-time. From a physicist's perspective, spin networks are "fantastically useful both as a basis for the states of quantum geometry and as a computational tool" [Ma]. From a mathematician's perspective, spin networks are fundamentally based in linear algebra, therefore offering an interesting way to diagram and carry out the often messy problems in that very wide field of study.

In [Cv], Pedrag Cvitanovic uses doodles—he called them "birdtracks"—inspired by Feynman's and modeled after Penrose's to explore various questions in Lie group theory. (See Figure A.) He so dubbed his diagrams when a non-mathematician friend, having seen his blackboards filled with diagrammatic scribble, exclaimed "What are all these bird-tracks?" Cvitanovic uses his diagrams in a more complicated setting than explored in this

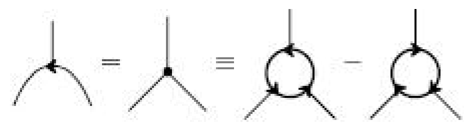


Figure 1. Identities with Cvitanovic' birdtracks.

paper, namely, Lie group theory and particularly the derivation of the classical and exceptional Lie algebras. However, like the diagrams in this paper, he makes the important distinction that his birdtracks, unlike Feynman's diagrams, are not simply a mnemonic aid, but a method of calculation. Proofs and derivations may be carried out, in Cvitanovic' paper as in this one, completely in doodles.

Faced with the calculations that fill books like Cvitanovic' or papers like this one, a knot theorist might feel most at home. The diagrams consist essentially of strands floating in the plane which, although they conceal some very cumbersome underlying math, may be manipulated simply and elegantly using topological rules. For example, the binor identity of Penrose's spin networks, in Figure A, is a reformulation of the Kauffmann skein relation common in knot polynomials. Also, since the diagrams are topologically invariant (a con-

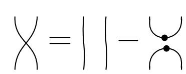


Figure 2. Binor Identity.

cept we will discuss in more detail later), they can be manipulated much like knots under ambient isotopy.

B. DISCOVERING PROOF WITH DIAGRAMS

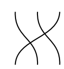
We attempt to present a definition of the diagrams which is rigorous enough to allow our proofs to stand alone. At the very least, diagrams provide a way of sketching a proof that is elegant, easy to understand, and often very revealing about the underlying nature of the problem. The diagrams provide a way of giving non-traditional, but rigorous, proof. We only hope that these diagrams may gain acceptance and become installed enough in the mathematical repertoire to be accepted by themselves as proof.

II. BACKGROUND

A. PERMUTATIONS

To begin our discussion, let's review a widely used concept in mathematics: permutations.

A permutation is simply a mathematical reordering. For example, we might permute the numbers $(1, 2, 3)$ to $(3, 1, 2)$. This may be written in *cyclic notation* as (132). Pictorially, you might think of it as the following diagram, read bottom to top:



There are actually $3! = 6$ ways to permute 3 elements. We represent all such permutations with S_n, the set of all permutations on n things, whose size is $n!$.

A permutation's *signature* is also useful. Think of the permutation as a series of swaps (or transpositions). In our (132) example, we first swap 1 and 2, then 2 and 3. Because there were 2 swaps, we call (132) an *even* permutation. (12) is an example of an *odd* permutation. This signature, denoted $\mathrm{sgn}(\cdot)$, is $+1$ if it's even and -1 if odd. In the diagrammatic notation, notice that the signature may be found by simply counting the number of crossings.

We may also name a permutation and refer to an individual element. For example, letting our first example be named $\sigma = (132)$, then $\sigma(1) = 3$. This concepts will play a big role in our discussions of linear algebra and trace algebras later in the paper. Also, the permutation diagrams just shown will blend naturally with the diagrams used in this paper.

B. LINEAR ALGEBRA PRELIMINARIES

Let's now review some key concepts in linear algebra. We begin with two important properties of a matrix, the *trace* and the *determinant*.

The trace of some $n \times n$ matrix A, $\mathsf{tr}(A)$, is simply the sum of its diagonal entries. For example,

$$\mathsf{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d. \tag{2-1}$$

The definition of the determinant for an $n \times n$ matrix $A = (a_{i,j})$ is

$$\mathsf{det}(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)}, \tag{2-2}$$

Recall the determinant of a 2×2 matrix,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a 3×3 matrix, the definition produces

$$\begin{aligned} \mathsf{det}(A) = {} & a_{1,1}a_{2,2}a_{3,3} + a_{1,3}a_{2,1}a_{3,2} + a_{1,2}a_{2,3}a_{3,1} \\ & -a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{aligned}$$

However, with $n!$ summands, for larger matrices this definition quickly leads to clumsy and inefficient calculations.

Another important concept in linear algebra is the notion of a *minor*, which is the determinant of any smaller square block of the matrix, or submatrix. So, for the following 4×4 matrix,

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix},$$

there is, for example, the 2×2 minor $\mathsf{det} \begin{pmatrix} c & d \\ g & h \end{pmatrix}$. One may think of a minor as the result of "crossing out" rows and columns of a matrix, then taking the determinant. In our example, we have crossed out rows 3 and 4 and columns 1 and 2. A *complemental minor* is then the minor resulting from the intersection of these crossed out rows and columns. In our example, the complemental minor is the 2×2 minor $\begin{pmatrix} i & j \\ m & n \end{pmatrix}$.

This leads to the idea of a *cofactor*. A cofactor $C_{i,j}$ is defined as

$$C_{i,j} = (-1)^{i+j} m_{i,j} \tag{2-3}$$

where $m_{i,j}$ is the minor resulting from crossing out row i and column j. So for our 4×4 matrix A, we have

$$C_{1,2} = -\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix}.$$

Using cofactors, determinants may be calculated by expanding a determinant along a row or column. For example, along row i of the matrix $A = (a_{i,j})$,

$$\mathsf{det}(A) = \sum_{j=1}^{n} a_{i,j} C_{i,j}.$$

This formula is often used when explicitly taking the determinant by hand, and is called "Laplace expansion."

A final concept, and one vital to our understanding of the Jacobi theorem we discuss in this paper, is the *adjugate*. The adjugate of a matrix is simply the transpose of a matrix consisting of all the cofactors. That is, given a matrix $M = (a_{i,j})$, for the adjugate of M we have $M_{\mathsf{adj}} = (C_{j,i})$. The adjugate makes frequent appearances in linear algebra. One identity for calculating the inverse, which involves the adjugate, is:

$$A_{\mathsf{adj}} \frac{1}{\mathsf{det}(A)} = A^{-1}.$$

Another well-known identity is

$$\mathsf{det}(A_{\mathsf{adj}}) = \mathsf{det}(A)^{n-1}, \qquad \text{(2-4)}$$

which is in fact a special case of a theorem we will prove later in this paper.

C. DODGSON'S CONDENSATION

The amateur mathematician Charles Dodgson, better known as the author Lewis Carroll, discovered an intriguing method for calculating the determinant. Dodgson called it his *Condensation Method*, and he describes his algorithm with four steps: Given an $n \times n$ matrix A, $n \geq 3$,

1. Arrange A, through row operations or otherwise, so that there are no zeroes in its $(n-2) \times (n-2)$ interior,
2. Condense A to make a second matrix, $(n-1) \times (n-1)$, which consists of the determinant of each connected 2×2 submatrix,
3. Condense this second matrix in the same way, but now dividing each term by the corresponding term in the interior of A,
4. Repeat this process of condensation as often as necessary, until the matrix is condensed to a single term, which will be the required value.

So for example, the matrix M equal to

$$\begin{vmatrix} -2 & -1 & -1 & -4 \\ -1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix}.$$

condenses to

$$\begin{vmatrix} 3 & -1 & 2 \\ -1 & -5 & 8 \\ 1 & 1 & -4 \end{vmatrix}$$

and again, to $\begin{vmatrix} 8 & -2 \\ -4 & 6 \end{vmatrix}$, whose determinant -8 is in fact the determinant of the original matrix M. [Dod]

Dodgson's method is not the most robust one known for calculating the determinant—in fact, it is useless unless you can eliminate all zeroes from the interior of the matrix—but it provides a curious look into the structure and nature of matrices and their determinants.

1. Jacobi Theorem

The Condensation Method makes use of a Jacobi determinant identity. This identity can be stated as follows:

Theorem 2.1: (Jacobi). *Given an $n \times n$ matrix M, choose any $m \times m$ minor A, its complement A', and the corresponding minor in the adjugate, A_{adj}. Then,*

$$\mathsf{det}(A_{\mathsf{adj}}) = \mathsf{det}(A')\mathsf{det}(M)^{m-1}. \tag{2-5}$$

Many identities in linear algebra are simply special cases of this theorem.

For example, let $m = n$. This leaves $\mathsf{det}(A') = 1$ and $\mathsf{det}(A_{\mathsf{adj}}) = \mathsf{det}(M_{\mathsf{adj}})$. We then have the equation

$$\mathsf{det}(M_{\mathsf{adj}}) = \mathsf{det}(M)^{n-1},$$

which has been mentioned already, as (2-4).

Dodgson's Condensation Method is itself a rearrangement of this formula.

2. Deriving Dodgson

Consider an $n \times n$ matrix $M = (a_{i,j})$:

$$M = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

Note that if we choose the 2×2 minor consisting of the outer four terms of this matrix, then $m = 2$, and we have the following:

$$A = \begin{pmatrix} a_{1,1} & a_{1,n} \\ a_{n,1} & a_{n,n} \end{pmatrix} \quad A' = \begin{pmatrix} a_{2,2} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix}$$

$$A_{\mathsf{adj}} = \begin{pmatrix} C_{1,1} & C_{n,1} \\ C_{1,n} & C_{n,n} \end{pmatrix} \qquad .$$

Now consider the terms of A_{adj} as the determinants of the southeast, northwest, southwest, and northeast $(n-1) \times (n-1)$ minors of M. We call these terms M_{SE}, M_{NW}, M_{SW}, and M_{NE}, respectively. Then

$$\mathsf{det}(A_{\mathsf{adj}}) = M_{NW}M_{SE} - M_{NE}M_{SW}.$$

Note that, for this purpose, we may ignore the associated signs of the cofactors because $C_{1,1}$ and $C_{n,n}$ will always be positive, and any negative sign on $C_{n,1}$ or $C_{1,n}$ will cancel out when taking the determinant.

Now let $M_C = \mathsf{det}(A')$, representing the determinant of the interior of the matrix, and we may rewrite the Jacobi theorem, for this special case of $m = 2$, as:

$$\mathsf{det}(M) = (M_{SE}M_{NW} - M_{SW}M_{NE})/M_C. \tag{2-6}$$

It can be shown that Equation (2-6) is not only a special case of Jacobi's theorem, it is actually Dodgson's Condensation Method.

To illustrate this, consider an arbitrary 4×4 matrix M,

$$M = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}.$$

Using Dodgson's Condensation as originally outlined, we may condense M to give:

$$\begin{pmatrix} af - be & bg - cf & ch - dg \\ ej - fi & fk - gj & gl - hk \\ in - jm & jo - kn & kp - lo \end{pmatrix},$$

which condenses to

$$\begin{pmatrix} \frac{(af-be)(fk-gj)-(bg-cf)(ej-fi)}{f} & \frac{(bg-cf)(gl-hk)-(ch-dg)(fk-gj)}{g} \\ \frac{(ej-fi)(jo-kn)-(fk-gj)(in-jm)}{j} & \frac{(fk-gj)(kp-lo)-(gl-hk)(jo-kn)}{k} \end{pmatrix}, \tag{2-7}$$

which condenses again to a very messy polynomial, the determinant. However, let us use (2-6) on M. We will use the formula recursively to determine the minors M_{SE}, M_{NW}, M_{SW}, and M_{NE}. By (2-6),

$$\begin{aligned} M_{NW} &= ((af - be)(fk - gj) - (bg - cf)(ej - fi))/f \\ M_{SE} &= ((fk - gj)(kp - lo) - (gl - hk)(jo - kn))/k \\ M_{NE} &= ((bg - cf)(gl - hk) - (ch - dg)(fk - gj))/g \\ M_{SW} &= ((ej - fi)(jo - kn) - (fk - gj)(in - jm))/j. \end{aligned}$$

These are, of course, the four terms in (2-7). And if we now apply (2-6) again, with $M_C = fk - gj$, we will obtain the same messy polynomial, and therefore the same value for the determinant, as we would using Dodgson's Condensation.

This is because Equation (2-6) is actually Dodgson's Condensation method in disguise, and it is the form used by much of the literature [Pr, AZ]. The Condensation Method recursively reduces the complicated problem of finding $\det(M)$ into the hand-calculable problem of finding many 2×2 determinants.

D. TRACE ALGEBRAS, FRICKE-VOGT

Because this paper also involves study of the *trace ring* on special linear matrix groups–i.e, invertible matrices of determinant 1, denoted $\mathrm{SL}(n, \mathbb{C})$–and specifically using diagrams to determine trace relations, we use this section to introduce some of the necessary concepts in this field. We have relied heavily on [Pet2, Gol] for a thorough exposition.

A *word* we define to be some combination of *letters*. That is, for the set of k letters $A_k = \{a_1, \ldots, a_k\}$, we have the set of words $\mathsf{W}(A_k)$, to include e, the empty word. We also consider inverses, writing this as $\bar{a} = a^{-1}$.

For example, for the set $A_3 = \{a, b, c\}$, the set of words $\mathsf{W}(A_3)$ will include ab, $abb\bar{b}aa$, etc. We construct the set of *reduced words* by adding the relations $a\bar{a} = \bar{a}a = e$. On the set of 3 letters A_3 as before, we might have, for example, the word $a^2\bar{c}ab$.

We call this set of reduced words in $\mathsf{W}(A_k)$ the *free group of rank k*, with identity e, and the operation of concatenation. We denote this

$$\mathsf{F}_k = \langle a_1, \ldots, a_k \rangle .$$

NOTE: The notation $\mathsf{W}(A_k)$ will always refer to all words, while the notation F_k will refer specifically to the set of reduced words.

One other basic concept we must introduce is a *polynomial ring*. Given a set S, the $\mathbb{C}$-polynomial ring $\mathbb{C}[S]$ is the set of polynomials with complex coefficients whose variables are taken from S. We are perhaps most familiar with $\mathbb{C}[x]$.

1. Introducing the trace

Now, let $G = \mathrm{SL}(n, \mathbb{C})$ and let $\rho : A_k \to G$, which extends to the group homomorphism $\mathsf{F}_k \to G$. In other words, ρ sends elements of the group F_k to the matrix group

G. The set of all such homomorphisms we call $\mathsf{Hom}(\mathsf{F}_k, G)$. These maps are called *representations*, and *representation theory* is an area of mathematics where abstract groups are studied under homomorphisms into linear matrix groups.

For this particular representation, recall the simple *trace* function discussed in "Linear Algebra Preliminaries." Consider the trace operator tr_ρ which gives a map from F_k to $\mathsf{Fun}(G^{\times k})$, the set of functions $f : G^{\times k} \to \mathbb{C}$. So, the trace operator tr_ρ is defined by

$$\mathsf{F}_k \to \mathsf{Fun}(G^{\times k}) \tag{2-8}$$
$$w(a_1, \ldots, a_k) \mapsto \mathsf{tr}(w(\rho(a_1), \ldots, \rho(a_k))). \tag{2-9}$$

So for example, the word $(a_1 a_2)$ maps to $\mathsf{tr}(\rho(a_1)\rho(a_2)) = \mathsf{tr}(A_1 A_2)$, which is a function taking two matrices to $\mathbb{C}$. And by extension, the trace also acts on $\mathbb{C}[\mathsf{F}_k]$, so that we have

$$\mathsf{tr}_\rho : \mathbb{C}[\mathsf{F}_k] \to \mathsf{Fun}(G^{\times k}).$$

We are essentially mapping the reduced word polynomial ring built on A_k into a trace on the words' representations. The image will be a polynomial ring sending representations to $\mathbb{C}$. This image denote with $\mathsf{T}\mathbb{C}_k^n$, and call it the *trace ring*.

The trace is a fundamental part of representation theory: it is a simple property of a matrix, but it encodes important information about the matrix, and hence, the structure of the group the matrices are representing. To develop this idea, we must first refine our idea of a trace ring.

Consider the kernel of the trace operator, $\ker(\mathsf{tr}_\rho)$, or specifically, what word polynomials transform to zero under the map in (2-9) for every ρ. For example, if v and w are cyclically equivalent words (the letters of one can be "rotated" to form the other, e.g. aba and a^2b), then $\mathsf{tr}_\rho(v) = \mathsf{tr}_\rho(w)$ by the properties of the trace. Therefore,

$$v - w \in \ker(\mathsf{tr}_\rho).$$

There are many other relations. If we think of removing these elements of the kernel from the word-polynomial ring, we create the quotient $\mathbb{C}[\mathsf{F}_k]/\ker(\mathsf{tr}_\rho)$ which must be then isomorphic to the trace ring $\mathsf{T}\mathbb{C}_k^n$. Figure 1 summarizes these relationships. The kernel is determined by recognizing these various *trace relations*. For example, let us compute $\mathsf{T}\mathbb{C}_1^2$

$$\begin{tikzcd} & \mathsf{T}\mathbb{C}^n_k \subset \mathsf{Fun}(G^{\times k}) \arrow[d, leftrightarrow, "\cong"] \\ \mathbb{C}[\mathsf{F}_k)] \arrow[ur, "\mathsf{tr}_\rho"] \arrow[r] & \frac{\mathbb{C}[\mathsf{F}_k)]}{\ker(\mathsf{tr}_\rho)} \end{tikzcd}$$

Figure 3. Algebraic construction of the trace ring.

for a single matrix $M = \rho(a)$ in $\mathrm{SL}(2, \mathbb{C})$. The image of tr_ρ is constructed from

$$\{\ldots, \mathsf{tr}(\bar{M}^2), \mathsf{tr}(\bar{M}), \mathsf{tr}(\mathbb{I}), \mathsf{tr}(M), \mathsf{tr}(M^2), \ldots\}.$$

However, the characteristic equation for M, restated using the Cayley-Hamilton Theorem, gives the relation

$$M^2 - \mathsf{tr}(M)M + \mathbb{I} = 0, \tag{2-10}$$

which, taking the trace, gives the relation

$$\mathsf{tr}(M^2) - \mathsf{tr}(M)^2 + 2 = 0.$$

Now, multiply (2-10) by $\bar{M}$ and take the trace to give

$$\mathsf{tr}(M) - \mathsf{tr}(\bar{M}) = 0$$

This gives two relations:

$$(a^2) - (a)^2 + 2 \in \ker(\mathsf{tr}_\rho), \;\; (a) - (\bar{a}) \in \ker(\mathsf{tr}_\rho)$$

Together, these two relations show that $\ker(\mathsf{tr}_\rho)$ contains enough relations to eliminate all elements of $\mathsf{T}\mathbb{C}^n_k$ except for $\mathsf{tr}(M)$–i.e., the trace of every word-polynomial in $\mathbb{C}[\mathsf{F}_1]$ can be written as a polynomial on just $\mathsf{tr}(M)$–so that we may write

$$\mathsf{T}\mathbb{C}^2_1 = \mathbb{C}[\mathsf{tr}(M)],$$

which is essentially the polynomial algebra on one variable $\mathbb{C}[x]$.

2. $\mathsf{T}\mathbb{C}_k^n$ and the invariant functions

We will now introduce some concepts from algebraic geometry to show that the invariant trace ring is equivalent to *all* invariant functions. First, we will show that the trace is invariant under the action of simultaneous conjugation. Then we will show the elements in the trace rings are precisely all of the invariant regular functions.

Simultaneous conjugation is a G-action on $G^{\times k}$ given by

$$g \cdot (\mathbf{x}_1, \ldots, \mathbf{x}_k) \ \mapsto \ (g\mathbf{x}_1 g^{-1}, g\mathbf{x}_2 g^{-1}, \ldots, g\mathbf{x}_k g^{-1}).$$

Recall our set of homomorphisms (all representations) which we denoted $\mathsf{Hom}(\mathsf{F}_k, G)$, and notice the correspondence

$$\rho \ \longleftrightarrow \ (\rho(A_1), \ldots, \rho(A_k))$$

which implies $\mathsf{Hom}(\mathsf{F}_k, G) \cong G^{\times k}$. So, applying simultaneous conjugation to $\mathsf{Hom}(\mathsf{F}_k, G)$ is identical to a simultaneous "change of basis" action on the matrix variables in $G^{\times k}$. We now see that the trace will be invariant under this action, since $\mathsf{tr}(A) = \mathsf{tr}(gAg^{-1})$. Therefore, all the functions in the trace ring $\mathsf{T}\mathbb{C}_k^n$ are invariant under simultaneous conjugation.

In algebraic geometry, a *variety* is the zero locus of a set of polynomials. For example, the locus of points satisfying the polynomial relation

$$\begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix} - 1 = 0$$

is the group $\mathrm{SL}(n, \mathbb{C})$, and so the group $G = \mathrm{SL}(n, \mathbb{C})$ is a variety.

The coordinate ring $\mathbb{C}[V]$ of a variety V is the set of polynomial functions which does not completely vanish (go to zero) on the variety. Such functions are named *regular* functions. For the variety $G = \mathrm{SL}(n, \mathbb{C})$, for example, the trace ring provides one such set of polynomials, since the trace does not go to zero if the determinant is 1.

Just as G is a variety, so is $G^{\times k}$. Then $\mathbb{C}[G^{\times k}] \cong \mathbb{C}[\mathsf{Hom}(\mathsf{F}_k, G)]$ are equivalent coordinate rings of equivalent algebraic varieties. Within the ring $\mathsf{Hom}(\mathsf{F}_k, G)$, the subring which is invariant under the G-action of simultaneous conjugation we denote $\mathbb{C}[\mathsf{Hom}(\mathsf{F}_k, G)]^G$. In order to construct this subring, we must choose only those elements of $\mathbb{C}[\mathsf{Hom}(\mathsf{F}_k, G)]$ which are invariant, throwing out the rest.

We would like to use some variety to construct a coordinate ring with identical structure to this ring of invariants. The quotient $G^{\times k}/G$ is not a suitable choice since the cosets (or orbits) of the G variety under the G-action are not always closed.

However, by using a construction called the *categorical quotient*–denoted $//G$–to remove enough "bad points" from $G^{\times k}$, we can ensure closure. The remaining, good points are called the *semistable points*, and they correspond to the *completely reducible* elements of $\mathsf{Hom}(\mathsf{F}_k, G)$. Now we may denote these good points of $G^{\times k}$ as $\mathsf{Hom}^{ss}(\mathsf{F}_k, G)$, and continue with our initial construction, obtaining the series of equivalent definitions

$$\mathfrak{X} \;\equiv\; \mathsf{Hom}^{ss}(\mathsf{F}_k, G)/G \;\equiv\; \mathsf{Hom}(\mathsf{F}_k, G)//G.$$

This is called the *G-character variety* of F_k, and its coordinate ring is then the desired $\mathbb{C}[\mathfrak{X}] \cong \mathbb{C}[\mathsf{Hom}(\mathsf{F}_k, G)]^G$.

It now only remains to demonstrate that $\mathsf{T}\mathbb{C}_k^n \cong \mathbb{C}[\mathfrak{X}]$ to show that the trace ring contains all invariant functions, our original goal. We state this as a theorem, from [Pro]:

Theorem 2.2: *Every* regular *function on $G^{\times k}$ which is invariant under simultaneous conjugation may be expressed as a polynomial on trace words. That is,*

$$\mathbb{C}[\mathsf{Hom}(\mathsf{F}_k, G)]^G \cong \mathsf{T}\mathbb{C}_k^n \cong \mathbb{C}[\mathfrak{X}].$$

Obviously $\mathsf{T}\mathbb{C}_k^n \subset \mathbb{C}[\mathsf{Hom}(\mathsf{F}_k, G)]^G$ since every trace function is a regular (does not vanish on the variety) and invariant (under simultaneous conjugation) function. We are obviously more concerned with the reverse direction, how to represent a regular invariant function as a trace function, but suffice it to say for now, it can be done.

Central to our questions about the trace ring should at this point be: how do we compute it? In general we would like to form a coordinate ring on some variables which generate the group (generators) and mod out unnecessary relations. However, the construction

$$\mathsf{T}\mathbb{C}_k^n \cong \mathbb{C}[\mathsf{F}_k)]/\ker(\mathsf{tr}_\rho)$$

is much *too* general. We want a minimal set of generators and an explicit set of relations. As we demonstrated with our simple example $\mathsf{T}\mathbb{C}_1^2$, one must show enough relations to eliminate all unnecessary generators until only a minimal set remains.

This study of the generators and relations of the trace ring is very important in the field of *invariant theory*. Finding sufficient generators and relations are easy, but finding a minimal set is not known beyond a few basic cases.

3. Fricke-Vogt

This discussion leads us directly to one such basic case, the classical Fricke-Vogt theorem.

Formally, we may state the theorem as follows [Gol]:

Theorem 2.3: (Fricke-Vogt). *Let $G = \mathrm{SL}(2,\mathbb{C})$ and let G act on $H = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ by simultaneous conjugation. Then, for every regular function $f : H \to \mathbb{C}$ satisfying*

$$f(\mathbf{x}_1, \mathbf{x}_2) = f(g\mathbf{x}_1 g^{-1}, g\mathbf{x}_2 g^{-1}) \text{ for all } g \in G,$$

there exists a polynomial $F(x, y, z) \in \mathbb{C}[x, y, z]$ such that

$$f(\mathbf{x}_1, \mathbf{x}_2) = F(\mathsf{tr}(\mathbf{x}_1), \mathsf{tr}(\mathbf{x}_2), \mathsf{tr}(\mathbf{x}_1\mathbf{x}_2)).$$

So for example, for the 2×2 matrices $A, B \in \mathrm{SL}(2,\mathbb{C})$, and generators $x = \mathsf{tr}(A)$, $y = \mathsf{tr}(B)$, and $z = \mathsf{tr}(AB)$,

$$\mathsf{tr}(ABA^{-1}B) = xyz - x^2 - z^2 + 2$$

which is a polynomial in the ring $\mathbb{C}[x, y, z]$.

4. Using diagrams to compute $\mathsf{T}\mathbb{C}_k^n$

The diagrams outlined in this paper provide, as we will see, a very concise way of representing the trace. Because of this natural marriage between the diagrams and trace polynomials, certain of the doodles which we call "trace diagrams" provide a powerful way of computing the trace ring.

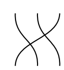
In fact, we believe that certain classes of diagrams may generate all of a certain associated class of relations in $\mathrm{SL}(n,\mathbb{C})$. In this paper, we explore one such class that generates a summation relation.

III. THE DIAGRAMS

We now turn to the central subject of this paper, the diagrams. Without attempting a complete exposition of diagrammatic methods, we will outline a few fundamentals as a primer to the technique. The reader may refer to [CFS, Cv, LP, Pet] for more comprehensive explanations.

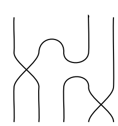
A. FUNDAMENTALS

In general, each diagram is a graph representing some underlying mathematics. In this paper, the diagrams will be read from bottom to top, inputs to outputs. We have already seen diagrams representing permutations, for example,



but they can be much more powerful than this.

Specifically, a diagram $D_{\mathrm{I}}^{\mathrm{O}}$, with number of inputs I and number of outputs O, represents some function $f : V^{\otimes I} \to V^{\otimes O}$ with $V = \mathbb{C}^n$. For example,



represents a map $V^{\otimes 5} \to V^{\otimes 3}$. The use of tensor products dates back to Penrose's spin networks, and are not only essential to the tensor calculus common in quantum calculations, but essential to allow the diagrams to remain multilinear maps.

To evaluate a diagram like the one above, we may decompose it into simpler pieces. We begin by defining the most basic pieces, and progress to show that from them we can construct any diagram.

Given $v, w \in V = \mathbb{C}^n$ and the standard basis vectors $\hat{e}_i$, $i \in N = \{1, \ldots, n\}$ for V, we have the basic maps:

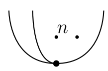
$$: v \mapsto v$$

$$: 1 \mapsto \hat{e}_1 \otimes \hat{e}_1 + \cdots + \hat{e}_n \otimes \hat{e}_n$$

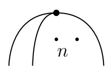
$$: v \otimes w \mapsto v \cdot w \text{ (inner product)}$$

$$: v \otimes w \mapsto w \otimes v$$

We also define two maps with nodes:



$$: 1 \mapsto \sum_{\sigma \in S_n} \text{sgn}(\sigma) \hat{e}_{\sigma(1)} \otimes \hat{e}_{\sigma(2)} \otimes \cdots \otimes \hat{e}_{\sigma(n)}$$



$$: u_1 \otimes u_2 \otimes \cdots \otimes u_n \mapsto \det(u_1 u_2 \cdots u_n)$$

Specific to $\mathbb{C}^2$, for example, these maps are:

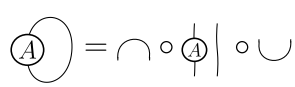
$$: 1 \mapsto \hat{e}_1 \otimes \hat{e}_2 - \hat{e}_2 \otimes \hat{e}_1$$

$$: v \otimes w \mapsto \mathsf{det}(v\, w)$$

As you may have noticed, because we are dealing exclusively with square matrices, we must also enforce the rule that within a diagram the degree of each vertex must equal n for $V = \mathbb{C}^n$.

B. SIMPLE APPLICATIONS

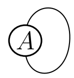
Given some $n \times n$ matrix $A = (a_{i,j})$, consider first the following very simple diagram and its decomposition into a series of basic maps:

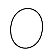


which leads to the mappings,

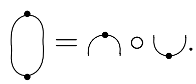
$$\begin{aligned} 1 &\longmapsto \hat{e}_1 \otimes \hat{e}_1 + \cdots + \hat{e}_n \otimes \hat{e}_n \\ &\longmapsto A\hat{e}_1 \otimes \hat{e}_1 + \cdots + A\hat{e}_n \otimes \hat{e}_n \\ &\longmapsto \sum_{i=1}^{n} a_{i,i} = \mathsf{tr}(A). \end{aligned}$$

Notice that we use tensor products between strands, and matrix multiplication within a strand. We have now verified the following:

Definition 3.1: *For any $n \times n$ matrix A,*  $= \mathsf{tr}(A)$.

Notice that a circle represents $\dim(V)$ since  $= \mathsf{tr}(\mathbb{I}) = n$.

Now, consider the following simple diagram and its decomposition into a series of basic maps:

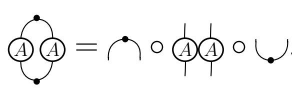


which leads to the mappings,

$$1 \longmapsto \hat{e}_1 \otimes \hat{e}_2 - \hat{e}_2 \otimes \hat{e}_1 \longmapsto \mathsf{det}\left(\hat{e}_1\ \hat{e}_2\right) - \mathsf{det}\left(\hat{e}_2\ \hat{e}_1\right) = 1 - (-1) = 2.$$

This is $2!$, and in fact, any composition of the noded cup and cap like this will result in the constant $n!$.

If we introduce a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ into the diagram, we have



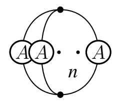
(3-1)

This leads to the mappings

$$\begin{aligned} 1 &\longmapsto \hat{e}_1 \otimes \hat{e}_2 - \hat{e}_2 \otimes \hat{e}_1 \\ &\longmapsto A\hat{e}_1 \otimes A\hat{e}_2 - A\hat{e}_2 \otimes A\hat{e}_1 \\ &\longmapsto \mathsf{det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \mathsf{det}\begin{pmatrix} b & a \\ d & c \end{pmatrix} = ad - bc - (bc - ad) = 2\,\mathsf{det}(A). \end{aligned}$$

Since a loop represents $n!$, and (3-1) consists of a loop with a matrix in $\mathbb{C}^2$, the factor of 2 on the determinant is what we might expect.

In general, we may say,

Proposition 3.2: *For an $n \times n$ matrix A,*  $= n!\,\mathsf{det}(A)$.

This leads us to our discussion of more complicated applications of the most basic diagrams.

C. MORE DEFINITIONS

We will now introduce a few more complicated diagrams and define them using the simpler diagrams already given.

First, because we will be working with basis vectors $\hat{e}_i$ for $i \in N = \{1, \ldots, n\}$, at this point it is convenient to develop a notation for the ordered sets of input and output basis vector indices. For the following definitions, assume we are given some $n \times n$ matrix A, and let $N = \{1, 2, \ldots, n\}$. Then choose $1 \leq m < n$, to construct the ordered subset

$$\alpha = (\alpha_1, \ldots, \alpha_m), \quad \alpha_i \in N,$$

and its complement, with $k = n - m$,

$$\beta = (\beta_1, \ldots, \beta_k), \quad \beta_i \in N \setminus \{\alpha_i\}.$$

We also introduce the following notation: given the two complemental ordered sets α and β, or permutations of either of these sets,

$$\mathrm{sgn}(\alpha|\beta)$$

represents the signature of the permutation from N to some ordered set

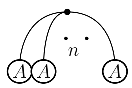
$$N' = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k).$$

Another important concept is the topological invariance of diagrams. For example, the following two diagrams are equivalent:

$$= \;|\;.$$

Proof of this concept is beyond the scope of this paper, but the reader may refer to [Pet] for further information.

Now we give several basic propositions which become useful in creating and evaluating generalized diagrams.

Proposition 3.3: 

$= \mathsf{det}(A)$ 

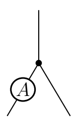
Proof. Evaluate the diagram on arbitrary vectors $u_1, \ldots, u_n \in V = \mathbb{C}^n$, giving the map:

$$Au_1 \otimes \cdots \otimes Au_n \longmapsto \mathsf{det}(Au_1 \ \cdots \ Au_n) = \mathsf{det}(A)\mathsf{det}(u_1 \ \cdots \ u_n),$$

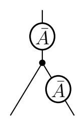
which can be written as the desired result. $\square$

Corollary 3.4: *Moving a matrix through a node distributes its inverse to the other strands and introduces a factor of the determinant.*

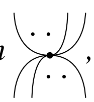
For example,

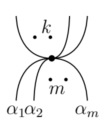


$=$



$\mathsf{det}(A).$

Proposition 3.5: *The diagram* *, with inputs $\{\hat{e}_{\alpha_i}\}$, is the following map:*

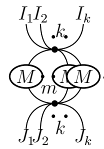


$$: \hat{e}_{\alpha_1} \otimes \cdots \otimes \hat{e}_{\alpha_m} \longmapsto \sum_{\sigma \in S_\beta} \operatorname{sgn}(\alpha|\sigma) \hat{e}_{\sigma(k)} \otimes \cdots \otimes \hat{e}_{\sigma(1)}.$$

A proof of this diagrammatic identity is included in the appendix. Also, the reader is referred to [Pet3] for detailed proofs of this and similar diagrams.

Using some of these concepts, we may also diagram any minor (determinant of a submatrix).

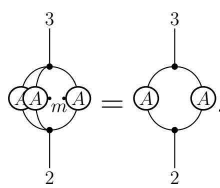
Proposition 3.6: *Given an $n \times n$ matrix M, the $m \times m$ minor $A_{I,J}$ with $|I| = |J| = k = n - m$ may be represented by the following diagram with inputs $\{\hat{e}_J\}$, outputs $\{\hat{e}_I\}$:*

$$\mathsf{det}(A_{I,J}) = m!$$ 

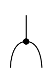
Note that Definition 3.6 includes the case $k = 1$. As an example, for the 3×3 matrix,

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

consider the diagram with $I = \{2\}$, $J = \{3\}$, and $m = 2$



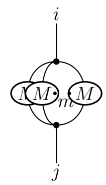
(3-2)

In the case of $n = 3$, the diagram  may be evaluated as the cross product of its input vectors. So, for example, (3-2) may be evaluated in the following way:

$$\begin{aligned}
\hat{e}_2 &\longmapsto \hat{e}_1 \otimes \hat{e}_3 - \hat{e}_3 \otimes \hat{e}_1 \\
&\longmapsto A\hat{e}_1 \otimes A\hat{e}_3 - A\hat{e}_3 \otimes A\hat{e}_1 \\
&\longmapsto A\hat{e}_1 \times A\hat{e}_3 - A\hat{e}_3 \times A\hat{e}_1 = 2(A\hat{e}_1 \times A\hat{e}_3) \\
&\longmapsto 2\left(\hat{e}_1(di - fg) - \hat{e}_2(ai - cg) + \hat{e}_3(af - cd)\right) \\
&\longmapsto 2(af - cd) = -2C_{3,2}.
\end{aligned}$$

This leads to a general case, and leads us to another result of Proposition 3.6,

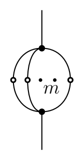
Corollary 3.7: *Given an $n \times n$ matrix M, $m = n - 1$, with input $\hat{e}_j$ and output $\hat{e}_i$, the* *following diagram represents the cofactor:*



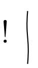
$$= (m!)\, C_{i,j}.$$

Another simple but useful result comes when we have no matrices on this type of diagram,

Corollary 3.8: (Bubble Identity). *The following diagram may be reduced:*



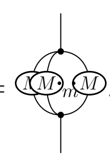
$$= m!$$



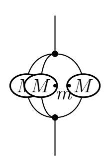
We will mention that since the adjugate matrix is simply the transpose of the matrix of cofactors, we might say for our matrix M,

Definition 3.9: *The adjugate* M_{adj} *of some matrix* M*, with* $m = n - 1$

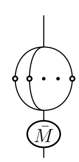
$$(n-1)!\, M_{\mathsf{adj}} =$$



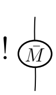
Proof. This arises from the following train of thought:



$$=$$



$$\mathsf{det}(M) = (n-1)!$$



$$\mathsf{det}(M),$$

which matches the equation

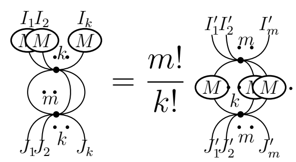
$$M_{\mathsf{adj}} = \mathsf{det}(M)\ \bar{M},$$

a well-known identity. □

This general diagram essentially represents a matrix, which given given input and output vectors represents a matrix entry (a cofactor). We will use this idea in the next section to diagram a proof of Jacobi's theorem.

Also, know that we may perform manipulations on the diagram for the minor. One such manipulation produces the following equality:

Proposition 3.10: *Given an $n \times n$ matrix M and some indices I, J of a minor, the following identity holds:*



Aside from detailed proof, notice that both sides of the equation contain the same number of matrices M, and the factors are somewhat intuitive given our knowledge of the diagram for a minor. The reader is referred to [Pet3] for a more detailed exposition of these types of manipulations.

We have now built a large enough diagrammatic vocabulary to demonstrate a proof of Jacobi's theorem.

IV. PROOF OF JACOBI'S THEOREM

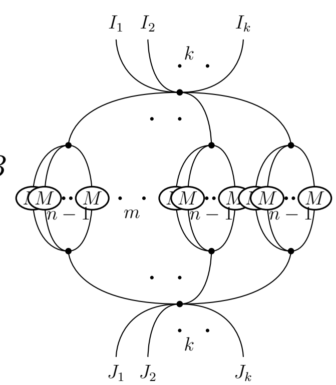
Here we will give a diagrammatic proof of the Jacobi theorem (2-5). The reader will see that the diagrams make the derivation of Jacobi's powerful formula so obvious the proof is, in some sense, trivial.

Recall that the Jacobi Theorem says that given an $n \times n$ matrix M, if we choose any $m \times m$ minor A, its complement A', and the corresponding minor in the adjugate, A_{adj}, then

$$\mathsf{det}(A_{\mathsf{adj}}) = \mathsf{det}(A')\mathsf{det}(M)^{m-1}$$

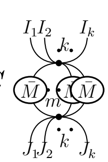
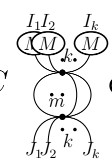
Proof. We will prove the theorem by first drawing the diagram for $\mathsf{det}(A_{\mathsf{adj}})$, then decomposing it into simpler parts equalling the right side of the equation. We will postpone evaluating the constant factors until the end, concerning ourselves chiefly with the diagram manipulation at work.

First, assume we have chosen some $m \times m$ minor $A = M_{I,J}$. Let $I' = N \setminus I$ and $J' = N \setminus J$. Now, consider the following diagram:

$\mathsf{det}(A_{\mathsf{adj}}) = B$  (4-1)

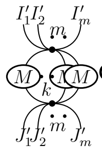
with some constant B. Notice we have used the diagram for the determinant of a minor, but instead of using M as our matrix, we are using the diagram for the adjugate. The result is the determinant of a minor in the adjugate, $\mathsf{det}(A_{\mathsf{adj}})$.

Now, shifting the matrices up out of the adjugate portion of the diagram and then to the top of the diagram using Corollary 3.4, "popping" any resulting bubbles using Corollary 3.8, we may reduce (4-1), to

BC  $\mathsf{det}(M)^m = BC$  $\mathsf{det}(M)^m \mathsf{det}(\bar{M})$.

with some factor C resulting from popping m bubbles.

Then, using Proposition (3.10), we may rearrange this to

BCD 

$$\mathsf{det}(M)^{m-1} = BCDE\ \mathsf{det}(A')\mathsf{det}(M)^{m-1},$$

with the factor D from 3.10 and E from the definition of a minor. We recognize this last expression as the diagram form of the right hand side of Jacobi's theorem, providing $BCDE = 1$.

The constant B is simply the constant for a minor with m copies of the constant for the adjugate. C resulted from popping m bubbles, so is also known. Using Proposition 3.10 introduces a constant D, and E comes from the constant for a complemental minor. So,

$$\begin{aligned} B &= \frac{1}{m!}\left(\frac{1}{(n-1)!}\right)^m \\ C &= ((n-1)!)^m \\ D &= \frac{m!}{k!} \\ E &= k! \end{aligned}$$

which cancel to 1. $\square$

Most interestingly, the classical proof of this theorem is quite messy and takes up several pages (see [RT] for an outline of the proof), whereas ours is completely carried out in diagrams and its essence could be represented in one line. It demonstrates the power of the diagrams. It also shows the wealth of information stored in the adjugate.

V. TRACE DIAGRAMS

We now turn our attention to the other subject of this paper, *trace rings*.

As already discussed, we are interested in computing the trace ring $\mathsf{T}\mathbb{C}_k^n$ for a given dimension n and rank k. As we have already seen, there is an elegant little diagram for the trace of a matrix: a circle with the matrix on it. We will call all such diagrams *n-trace diagrams*, and we will see they provide a rich machinery with which to compute trace relations.

This section begins by discussing existing knowledge in the $\mathrm{SL}(2,\mathbb{C})$ case for rank 3, and we give the diagrammatic form of the two most important relations, as outlined in [Pet4]. We then progress into $\mathrm{SL}(3,\mathbb{C})$, where fewer results are known due to the much greater difficulty of this case. In order to work with these more complicated 3-trace diagrams, we first develop an algorithm that allows us to diagrammatically simplify any 3-trace diagram into a trace polynomial. We lastly attempt to compute the summation relation for the rank 2 case.

NOTE ON NOTATION: Because the trace will arise frequently in our calculations in this section, we will often use the abbreviated notation $\mathtt{tr}(\mathbf{x}) = [\mathbf{x}]$. We also continue the $X^{-1} = \bar{X}$ convention.

A. $\mathrm{SL}(2,\mathbb{C})$

Recall that proof of the Fricke-Vogt theorem requires us to, among other things, give an explicit set of relations for a minimal set of generators.

In Theorem 2.3, the rank 2 case for $\mathrm{SL}(2,\mathbb{C})$ with matrices A and B, we have many candidate trace generators

$$[A], [B], [AB], [BA], [\bar{A}B], [ABAB], [A^2B], \ldots$$

but we know immediately that many of these are redundant by cyclic equivalency, and many more are unnecessary with the relations derived from the characteristic equation. The theorem states that in fact, we need only three: $[A]$, $[B]$, and $[AB]$.

We now seek to demonstrate a similar result for the rank 3 case, with matrices A, B, and C. Automatically removing cyclic equivalencies (ex., $[CAB] = [ABC]$, and knowing by the characteristic equation that we may remove squares and inverses, we are left with the following candidate generators:

$$[A], [B], [C], [AB], [AC], [BC], [ABC], [ACB].$$

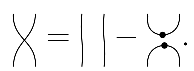
However, these last two generators are not independent. There is a summation relation and a product relation which together show we only need one or the other, not both. We will briefly show the diagrammatic method to derive the summation relation in this case.

To determine a relation diagrammatically, we can in general start with an entangled doodle and then, using well-defined rules, *un*tangle the doodle, leaving us with an equivalent expression—a relation! Let's first investigate one very important rule, the binor identity.

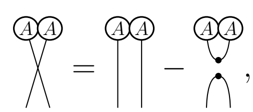
1. Binor identity

The most important of these well-defined rules is one which Penrose called the *binor identity*. As mentioned in the introduction, it is a specialization of an equation common in knot theory, especially in the computation of knot polynomials.

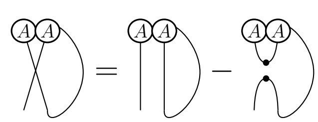
Identity 5.1: (Binor Identity). *In* $\mathrm{SL}(2,\mathbb{C})$*,*



Proof. We will show the binor identity is true by showing it is equivalent to the characteristic equation. First add the matrix $A \in \mathrm{SL}(2,\mathbb{C})$ to the top strands,

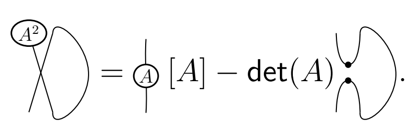


and close off one side of each diagram:



(5-1)

This is called "taking the trace." We may now simplify (5-1), using our given knowledge of diagram manipulation and topological invariance, to the statement



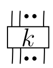
The first term may be untwisted, and the last term may be simplified to simply $\Big|$, giving by Cayley-Hamilton the characteristic equation

$$A^2 = A[A] - \mathsf{det}(A)\mathbb{I}.$$

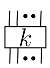
$\square$

Recall during our previous discussions where we introduced a diagrammatic notation for permutations. The *anti-symmetrizer* is the sum of all permutations with their associated signature. A similar concept has already been discussed in the introductory section on diagrams, but we introduce a new notation here.

Definition 5.2: *The* anti-symmetrizer *on k vertices with inputs $u_i \in V = \mathbb{C}^n$ is*



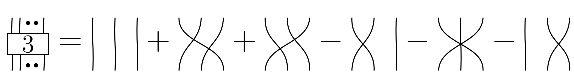
$$= a_1 \otimes \cdots \otimes a_k \mapsto \sum_{\sigma \in \S_k} \mathrm{sgn}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}.$$

Some texts introduce a constant factor $k!$, but we omit this. Note that if $k > n$, then  $= 0$.

2. Summation relation

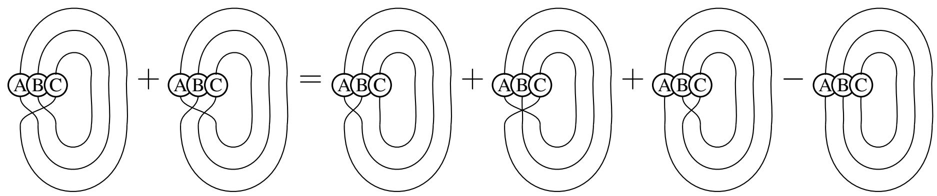
We now offer the diagrammatic derivation of the summation relation for rank 3. The reader should notice the elegance and simplicity which this method offers over the alternative, dealing with jumbles of trace arithmetic. Even more important than its evident simplicity, however, we suggest that the braided type of diagram which gives the summation relation in rank 3 for $\mathrm{SL}(2, \mathbb{C})$ is really the form, or prototype, of a whole family of diagrams which give the summation relation for rank k in $\mathrm{SL}(n, \mathbb{C})$.

Consider the anti-symmetrizer on 3 vertices,



$$= 0.$$

Now, add matrices A, B, and C, rearrange, and close off all strands to have the statement:



which is easily seen to be the equation

$$[ABC] + [ACB] = [C][AB] + [B][AC] + [A][BC] - [A][B][C]$$

the equation relating $[ABC]$ with $[ACB]$. [Pet4]

We may follow a similar procedure, although not with the anti-symmetrizer, to show that the product $[ABC][ACB]$ may also be expressed in simpler terms. However, we would prefer a less brute-force method than expanding the anti-symmetrizer. We look to extend this method from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SL}(n,\mathbb{C})$.

B. $\mathrm{SL}(3,\mathbb{C})$

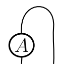
Trace calculations in $\mathrm{SL}(3,\mathbb{C})$ become disproportionately hard, very quickly. For the trace rings $\mathsf{T}\mathbb{C}_k^n$, the product relation which for $n = 2$, $k = 3$ takes up a couple lines, in $n = 3$, $k = 2$ takes up the better part of a page (see [Law]). In fact, entire journal articles have been written on the computation of this relation in recent years. Knowing the simplifying effect diagrams have on the problems which suit them, we hope that we may achieve some interesting results in the $\mathrm{SL}(3,\mathbb{C})$ case.

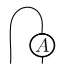
The good news is, we do (sort of)! First, since as we mentioned $\mathrm{SL}(3,\mathbb{C})$ is hard, we must first develop more robust ways of simplifying a diagram than simply the binor identity applied repeatedly. We present an algorithm to reduce any 3-trace diagram to a trace polynomial. Second, we apply the algorithm to a diagram which encodes the summation relation for $\mathrm{SL}(3,\mathbb{C})$, and present our partial results.

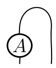
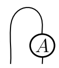
1. 3-trace diagrams, directed graphs

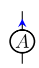
To review, each 3-trace diagram consists of nodes and edges where each node has an index of 3.

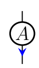
We must introduce one almost indispensable property of diagrams that up to this we've been able to ignore: the diagrams must be directed (in the graph sense) in order for their function to be well-defined. For example, consider the following two diagrams with arbitrary inputs $u, v \in V = \mathbb{C}^n$ and an arbitrary $n \times n$ matrix A:


$$= (Au) \cdot v$$

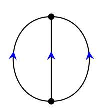

$$= u \cdot (Av).$$

We notice that although  and  are identical under the diagrams' purported topological invariance, they produce different functions: $u \cdot (Av) = (A^T u) \cdot v \neq (Au) \cdot v$. We may solve this, however, by requiring the edge on this graph to be directed, with a reverse direction forcing a transpose:

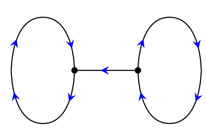

$$: v \mapsto Av$$


$$: v \mapsto A^T v$$

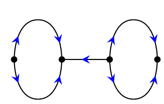
Since each edge must have a single direction, a little doodling will reveal that this requirement has forced another rule for the diagrams: every vertex must be either a *source* or *sink*. For example, note the direction of the arrows emanating from each vertex in the following diagram:



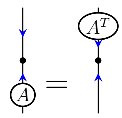
We have one more obstacle, however. What if it is impossible for a diagram to have vertices which are either sources or sinks? For example,



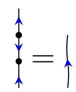
In this case, we introduce a temporary 2-node along the troublesome edge which is also required to be a source or sink:



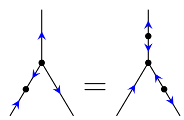
This node performs no mapping, only redirects arrows. However, notice that sliding a matrix through a 2-node will produce its transpose.



Also, notice that two adjacent 2-nodes on the same edge will cancel:



Lastly, notice that a 2-node may distribute itself through a vertex without changing the properties of the diagram, although this is not usually very useful.



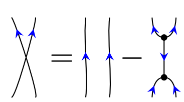
In general, the whole idea of the directed graph is a bookkeeping nuisance, which is sometimes necessary but may often be overlooked (as we have been doing until now). Unfortunately in this section, we must often be careful to keep track of our 2-nodes and transposes, or our result will be incorrect.

2. Identities

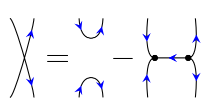
In this section we present basic identities for 3-trace diagrams.

The first identity we have already introduced, for $V = \mathbb{C}^2$, as the binor identity, and show here its $V = \mathbb{C}^3$ forms, taken from [Pet].

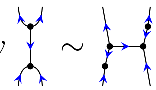
Identity 5.3: (Binor Identity).

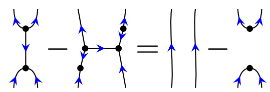


Identity 5.4: (Rotated Binor Identity).



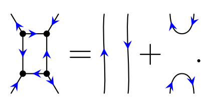
We notice immediately that the two forms are both equal to a crossing, and in fact, if we could somehow force the diagrams to have the same directions of inputs and outputs, we could form some equivalence. Using the 2-nodes earlier introduced,

Identity 5.5: (Binor Equivalency). *The equivalency*  *produces the following identity:*

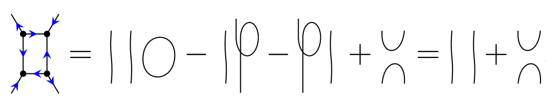


The next identity will also be of extreme importance, as it reduces a diagram with four nodes into two with none.

Identity 5.6: (Wheel Identity).



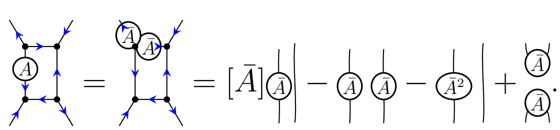
Proof. Without regard to arrows (we can temporarily ignore them), we have the following derivation by repeatedly applying the binor identity:



$\square$

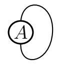
If there is a matrix in the center, we may still decompose the wheel diagram in a similar fashion, giving:

Identity 5.7: (Wheel with Matrix).

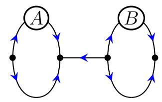


We must stress the importance of these binor and wheel identities: they reduce diagrams with crossings and nodes to ones without. Applying these identities repeatedly to a diagram will eventually reduce it to one of a simple, known form. We list three simple forms.

Proposition 5.8: *Simple known forms are as follows:*

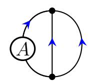
Type I (Loop). 

$= \mathsf{tr}(A) = [A]$.

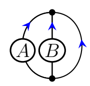
Type II (Barbell). 

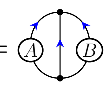
$= \mathsf{tr}(AB) - \mathsf{tr}(A^T B) = [AB] - [A^T B]$.

Type III (Double loop:).

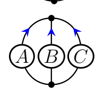


$= 2[A]$



$=$ 

$= [A][B] - [AB]$



$$= [A][B][C] + [ABC] + [ACB] - [A][BC] - [B][AC] - [C][AB].$$

Proof. The form I we have already defined, form II may be easily evaluated with the binor identity, and form III is derived in [Pet]. Notice that by the binor equivalency relation, the barbell can be transformed into the double loop. □

3. Algorithm for reducing any diagram

We have already seen that directly computing diagrams is not the best method. In fact, doing so completely defeats the purpose of the diagrams, which is to avoid cumbersome algebra as much as possible.

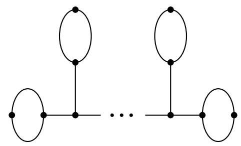
However, we need a systematic way to reduce unclear diagrams into known forms. Is it even possible to do this? This paper proposes that, at least for dimension 3, it is.

Theorem 5.9: *Any closed diagram in* $\mathrm{SL}(3, \mathbb{C})$ *containing matrices* A_i, $i = 1, 2, \ldots$, *may be expressed as a trace polynomial on* A_i, $\bar{A}_i$ *and* A_i^T.

As evidence, we present an algorithm which uses diagrammatic techniques to reduce any trace diagram to a "known form."

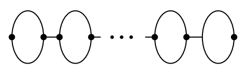
Algorithm 5.10: *Given some closed diagram, we may reduce it to a trace polynomial as follows:*

1. *Use the binor identity equivalence relation (Identity 5.5) to reduce the diagram to one of this form:*

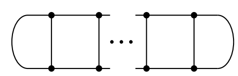


2. *Move all matrices onto the hanging "lollipops," introducing the inverse as necessary.*

3. *Use the binor identity (Identity 5.3) to transform each lollipop into a bubble on the line:*



4. *Use the binor identity on each bubble to form a ladder:*



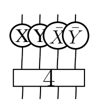
5. *Use the wheel identity (Identity 5.6) until the diagram is of form I or III.*

Notice that in each step using the binor equivalency relation, we leave a trail of simpler diagrams which must be recursively run through the algorithm themselves. For a demonstration of the algorithm, we will again turn to the summation relation for a trace ring of rank 2, this time in $\mathrm{SL}(3, \mathbb{C})$.

4. Summation Relation

In $\mathrm{SL}(3, \mathbb{C})$ for rank 2, the summation relation makes the terms $[XY\bar{X}\bar{Y}]$ and $[YX\bar{Y}\bar{X}]$ redundant, which we know from Sean Lawton's work in [Law].

One way to achieve this is by expanding the anti-symmetrizer. That is, expanding



$$= 0.$$

We expand, add matrices and take the trace four times, yielding the expression

$$
\begin{aligned}
[XY\bar{X}\bar{Y}] + [YX\bar{Y}\bar{X}] = {} & [X][Y][\bar{X}][\bar{Y}] + [X][\bar{X}] + [Y][\bar{Y}] \\
& + [\bar{X}\bar{Y}][XY] + [X\bar{Y}][\bar{X}Y] - [X][Y][\bar{X}\bar{Y}] \\
& - [X][\bar{Y}][\bar{X}Y] - [\bar{X}][\bar{Y}][XY] - [\bar{X}][Y][X\bar{Y}] - 3.
\end{aligned}
$$

The full expansion may be found in Appendix B. This is, in fact, the summation relation for the rank 2 case in $\mathrm{SL}(3, \mathbb{C})$.

However, we wish to achieve the same result by using a diagram in the braid-family. Consider the simple trace diagram and its expansion using the basic binor identity in Figure 4. We name each of the eight terms in Figure 4 as D_i. We evaluate the diagram as follows,

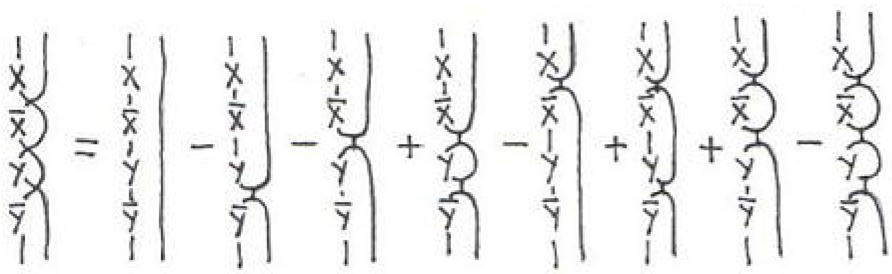


Figure 4. Trace expansion.

where we temporarily postpone evaluating the eighth term, D_8:

$$
\begin{aligned}
&= 6 - 6 - 6 + 3 + [Y][\bar{Y}] - 6 + 3 + \\
&\quad [\bar{X}Y][X\bar{Y}] - 3 + [X][\bar{X}] - D_8 \\
&= [Y][\bar{Y}] + [X][\bar{X}] + [\bar{X}Y][X\bar{Y}] - D_8
\end{aligned}
$$

The term D_8 is a bit trickier. We give a decomposition of this term in Appendix B: a knot theorist may find the equations there very similar to computing a bracket polynomial by hand. The process is straightforward, but tedious. Because the barbell form introduces the transpose, the term D_8 will contain some terms with the transpose. These are unwanted, as they are not invariant under the group action, and therefore have no place in the Fricke-Vogt theorem. A disadvantage of our algorithm for reducing diagrams is that it introduces these traces. Consider, however, the eighth term expressed as

$$D_8 = D_8^* + D_8^T$$

where D_8^* represents all the terms without the transpose, and D_8^T represents the terms with a tranpose.

Now, combining terms from all 8 parts of the expansion, we have the following partial result:

$$\begin{aligned}[XY\bar{X}\bar{Y}] =& [YX\bar{Y}\bar{X}] + [\bar{X}Y][X\bar{Y}] - 2[\bar{X}][X\bar{Y}][Y] \\ &+ [X][\bar{X}] + 2[Y][\bar{Y}] - 6 + D_8^T.\end{aligned}$$

This is close to the actual relation we gave before. The missing terms may be in the term D_8^T. The tranposes in D_8^T may cancel out. Also, we may also try working in $\mathrm{SO}(3, \mathbb{C})$, the special orthogonal group, in which case $M^T = \bar{M}$.

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VI. LOOKING AHEAD

This concludes the main work of this thesis. In summary, we have offered a diagrammatic proof of a classical theorem in linear algebra and explored the power of trace diagrams in constructing the trace ring.

In this section we will give an overview, for the interested reader, of a curious result of Dodgson's Condensation which is (as of yet) totally unrelated to the diagrams. We will also give a brief synopsis of what we think are important problems in the diagrams' future.

A. ALTERNATING SIGN MATRICES

An interesting offshoot of Dodgson's Condensation Method is the study of the alternating sign matrices (ASMs). While we have not yet been able to find a diagrammatic way to approach the associated problems, it is an interesting diversion we felt appropriate to include since we have examined Condensation to such depth, and ASMs are the chief—possibly only—direct result of Dodgson's method.

Study of the Dodgson Condensation Method led Robbins and Rumsey (see [Br, BP]) to the alternating sign matrices. Notice what happens when we evaluate Dodgson's algorithm on an arbitrary matrix:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \longrightarrow \begin{vmatrix} ae-bd & bf-ce \\ dh-eg & ei-fh \end{vmatrix}$$

and, continuing the condensation and dividing by the interior element e,

$$= (1)aei + (-1)afh + (-1)bdi + (0)bdfhe^{-1} + (1)bfg + (1)cdh + (-1)ceg.$$

We can associate six of these terms with a certain matrix; for example, $(-1)bdi$ is associated with a matrix with a 1 in the position of b, d, and i in the original matrix. This is a permutation matrix, and its signature is the coefficient -1. Also, even the vanishing term can be associated with a matrix following similar rules, and with a -1 in the position of e

(for its negative exponent). That is,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

As it turns out, these seven matrices are also the seven possible *alternating sign matrices* of size $n = 3$.

An Alternating Sign Matrix (ASM) is a matrix consisting of $+1$s, -1s, and 0s, subject to the following rules:

1. Each row and column must sum to 1.

2. Successive entries of ± 1 must alternate in sign (beginning with "$+$").

For example,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In fact, the Dodgson Condensation Method, when carried out as above on a generalized $n \times n$ matrix $M = (m_{ij})$, gives a series of monomials of the form:

$$\prod_{i,j=1}^{n} m_{ij}^{A_{ij}},$$

where A_{ij} is the i, jth entry of an alternating sign matrix.

Pursuing pure curiosity, Robbins and Rumsey asked the question: How many $n \times n$ ASMs are there? Let A_n denote the number of ASMs of size $n \times n$. After enumerating the first several cases, Robbins and Rumsey encountered the sequence

$$A_n = 1, 2, 7, 42, 429, 7436, 218348, \ldots.$$

And after careful study of the combinatorial nature of this sequence, the duo made the following conjecture: [Br]

Conjecture 6.1: (The ASM conjecture). *The total number of $n \times n$ alternating sign matrices is*

$$A_n = A_{n+1,1} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Mills, Robbins, and Rumsey were unable to prove their conjecture, but it was eventually solved after the subject had been tied into widely varied fields of mathematics like plane partitions, Schur functions, and statistical mechanics.

For more information on this subject, refer to the excellent book [Br], or the articles [BP, Pr].

B. AFTER RANK 2

This paper has addressed the construction of the trace ring $\mathrm{T}\mathbb{C}_2^3$, and while in this particular case it is already "solved," we feel the diagrams offer an additional perspective that could more easily catapult its user into solutions for higher ranks, or even, higher dimension.

In summary, we offer the following areas of study we hope find solution in future works:

1. Diagram trace relations in $\mathrm{SL}(n, \mathbb{C})$ with $n \geq 4$ and beyond rank 3.
2. Diagrammatic methods for other areas of linear algebra (e.g., manipulating non-square matrices).

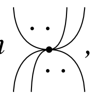
As for the second item, it seems that given such a vast theory as that surrounding linear algebra, being able to use the powerful techniques discussed briefly in this paper would be of immense usefulness. Linear algebra seeps into many other areas of mathematics, and so could these diagrams. What remains to be explored is a better understanding of the diagrams for $V = \mathbb{C}^n$, or a way to deal with $m \times n$ matrices, or even, a way to handle matrices of different sizes within the same diagram.

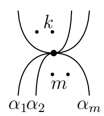
As a parting comment, much like books on knot theory, depicting diagrams seems to harangue any author attempting to use them. Much progress has been made to ease the digital drawing of knots, but little has been done for diagrams. Most of the diagrams in this paper are due to work Elisha Peterson has done in LaTeXwith the PGF/TiKZ package. However, the large number of specialized diagrams in this paper still managed to give the author grief, so we also hope that in the future, diagrammatic methods might attract not only doodle-minded mathematicians, but also some talented typesetters.

VII. APPENDICES

A. APPENDIX A

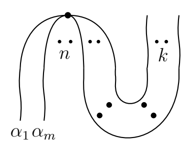
This section provides a proof that was omitted in the original diagram exposition. More information may be found in [Pet2].

Definition 7.1: *The diagram* *, with inputs* $\{\hat{e}_{\alpha_i}\}$*, is the following map:*



$$: \hat{e}_{\alpha_1} \otimes \cdots \otimes \hat{e}_{\alpha_m} \longmapsto \sum_{\sigma \in S_\beta} \operatorname{sgn}(\alpha|\sigma) \hat{e}_{\sigma(k)} \otimes \cdots \otimes \hat{e}_{\sigma(1)}.$$

Proof. Consider a rearrangement of the diagram as follows:



where we set $N = \{1, 2, \ldots, n\}$ and construct α as before. Note that we can rearrange the diagram like this because of the topological invariance of diagrams. This gives the mapping:

$$\longmapsto \hat{e}_{\alpha_1} \otimes \cdots \otimes \hat{e}_{\alpha_m} \otimes \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_k} \otimes \hat{e}_{i_k} \otimes \cdots \otimes \hat{e}_{i_1}$$

$$\longmapsto \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \mathsf{det} \left(\hat{e}_{\alpha_1} \cdots \hat{e}_{\alpha_m} \; \hat{e}_{i_1} \cdots \hat{e}_{i_k} \right) \otimes \hat{e}_{i_k} \otimes \ldots \otimes \hat{e}_{i_1}$$

Unless $\hat{e}_{\alpha_1} \ldots \hat{e}_{\alpha_m} \hat{e}_{i_1} \ldots \hat{e}_{i_k}$ form a basis for $\mathbb{R}^n$, this determinant will be zero, and the term will fall away. So, let $\beta = \{i_1 \ldots i_k\}$ for those i that produced a nonzero determinant. This shows that

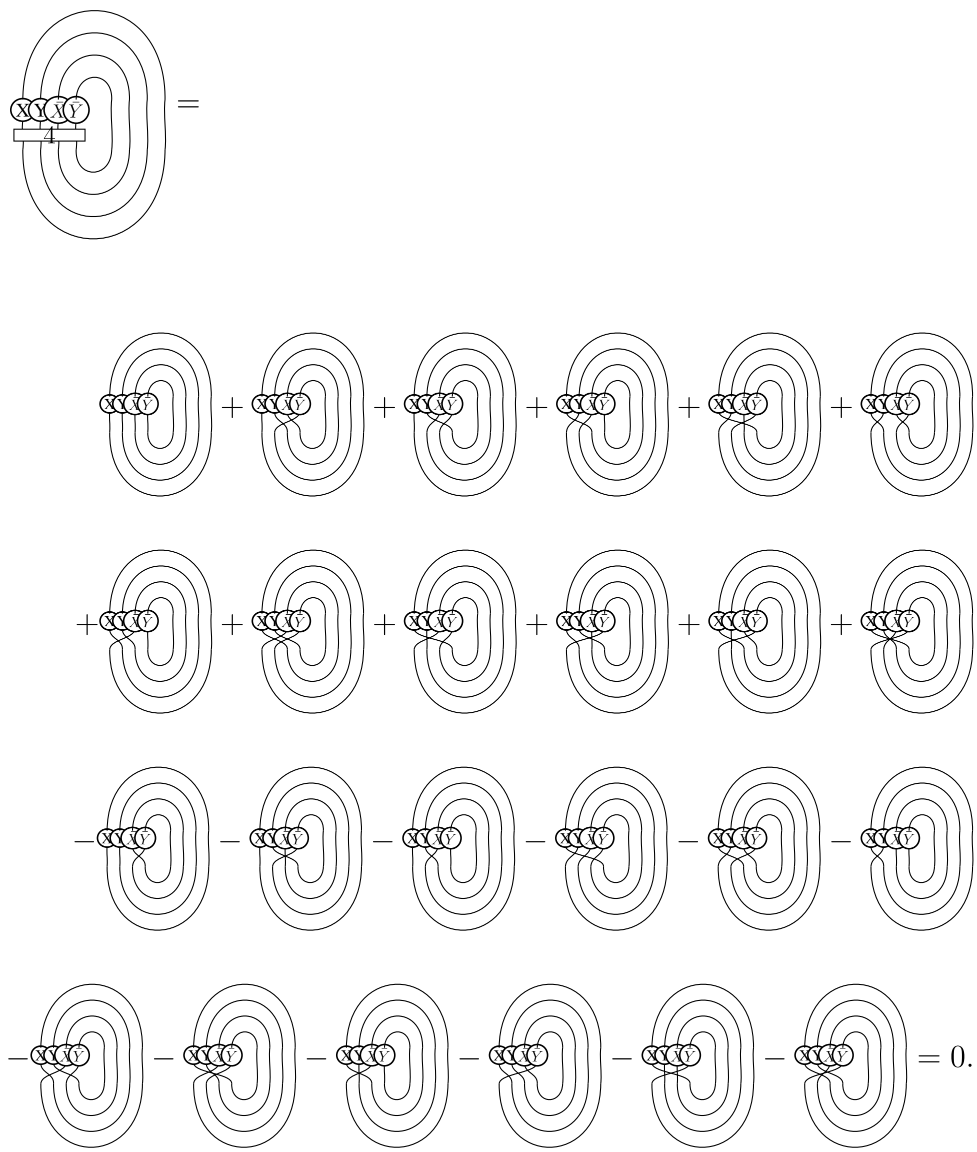
$$\beta = N \setminus \alpha_i, \quad |\beta| = n - m = k.$$

We also know the nonzero determinants must be ± 1, with the sign determined by $\operatorname{sgn}(\alpha|\sigma)$, $\sigma \in S_\beta$. $\square$

B. APPENDIX B

This appendix contains some summation relation calculations, first with the anti-symmetrizer, then with the braid-like diagram.

1. Anti-symmetrizer



2. Braid-like term

The braid-like derivation of the final term follows, and is partially handwritten.

The initial braid and its expansion is in Figure 5. After taking the trace twice, the

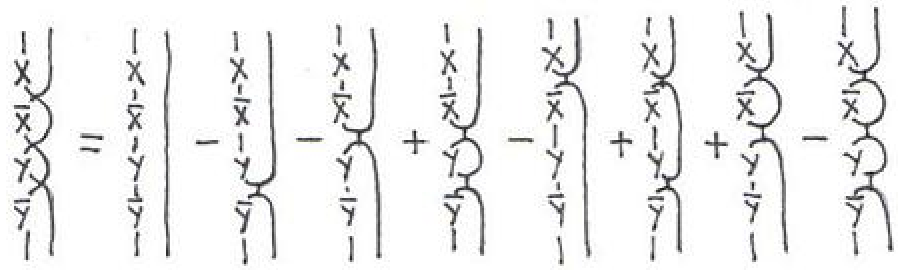


Figure 5. Braid expanded.

first 7 terms are easily evaluated using basic trace identities and the wheel identity, and are given in the paper. The eighth term, D_8, expanded using the Algorithm presented in this paper is in Figure 6. Using the identities in Figures 7, 8, and 9, we can reduce these labeled parts into a trace polynomial, given in Figure 10.

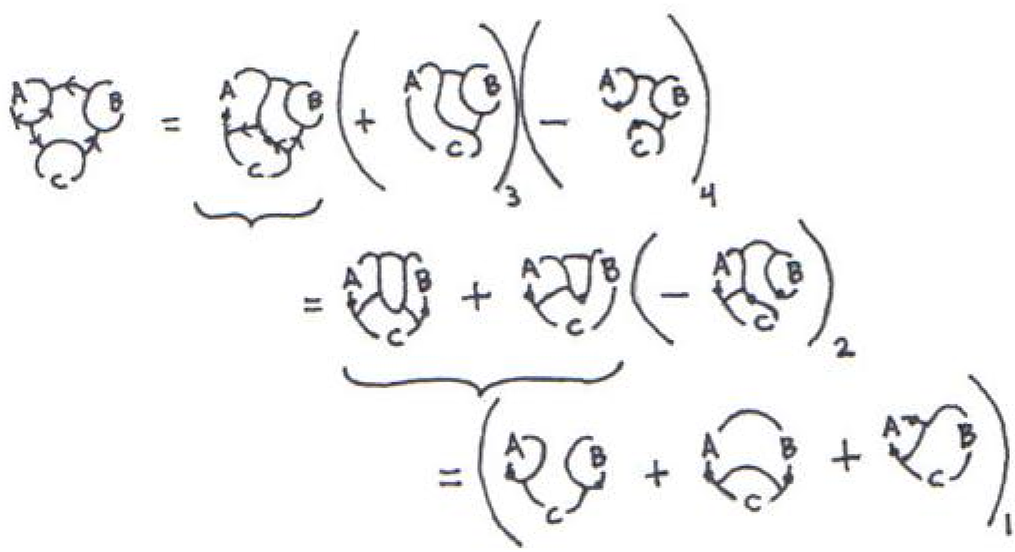


Figure 6. Term 8 expanded.

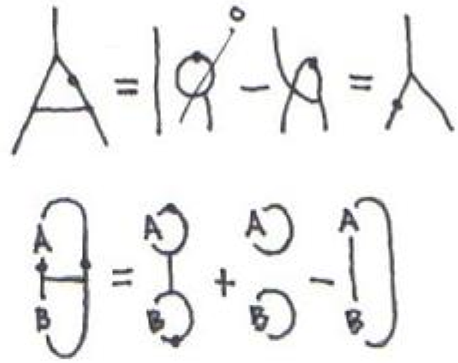


Figure 7. Two Identities.

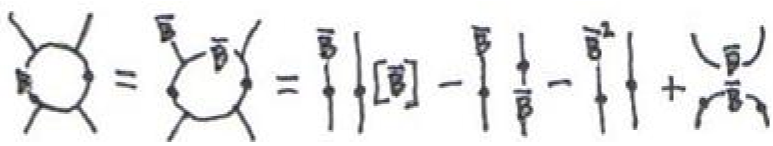


Figure 8. Wheel expanded with matrix and nodes.

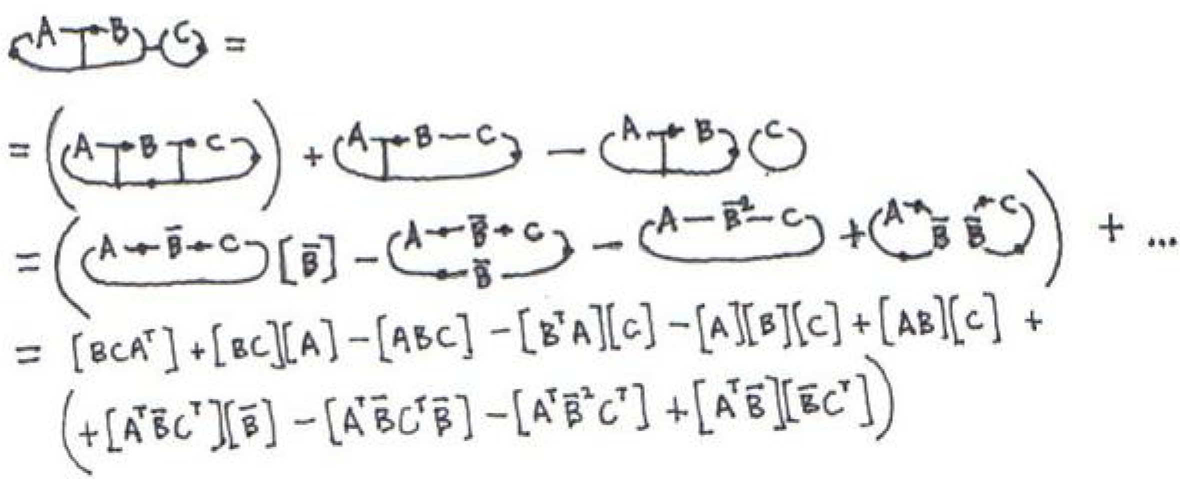


Figure 9. A tricky expansion used in terms 2 and 4.

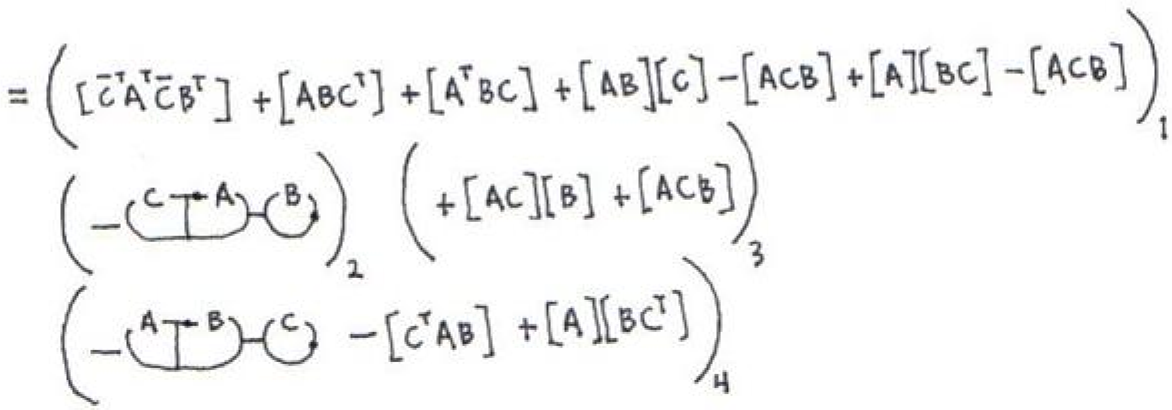


Figure 10. Expansion by parts.

This becomes the unsimplified polynomial:

$$
\begin{aligned}
=&([\bar{Y}^T\bar{X}^T\bar{Y}(X\bar{Y})^T] + [\bar{X}X\bar{Y}Y^T] + [\bar{X}^T X\bar{Y}Y] \\
&+ [\bar{X}X\bar{Y}][Y] - [\bar{X}YX\bar{Y}] + [\bar{X}][X\bar{Y}Y] - [\bar{X}YX\bar{Y}])_1 \\
&+ (-[\bar{X}X\bar{Y}Y^T] - [\bar{X}X\bar{Y}][Y] + [\bar{X}X\bar{Y}Y] \\
&+ [\bar{X}^T Y][X\bar{Y}] + [\bar{X}][X\bar{Y}][Y] \\
&- [\bar{X}Y][X\bar{Y}] - [Y^T X(X\bar{Y})^T][X] + [Y^T X(X\bar{Y})^T X] \\
&+ [Y^T X^2(X\bar{Y})^T] - [Y^T X][X(X\bar{Y})^T])_2 \\
&+ (+[\bar{X}Y][X\bar{Y}] + [\bar{X}YX\bar{Y}])_3 \\
&+ (-[X\bar{Y}Y\bar{X}^T] - [X\bar{Y}Y][\bar{X}] + [\bar{X}X\bar{Y}Y] \\
&+ [(X\bar{Y})^T\bar{X}][Y] + [\bar{X}][X\bar{Y}][Y] - [\bar{X}X\bar{Y}][Y] \\
&- [\bar{X}^T Y\bar{X}Y^T][Y\bar{X}] + [\bar{X}^T Y\bar{X}Y^T Y\bar{X}] + [\bar{X}^T Y\bar{X}^2 Y^T] \\
&- [\bar{X}^T Y\bar{X}][Y\bar{X}Y^T] - [Y^T\bar{X}X\bar{Y}] + [\bar{X}][X\bar{Y}Y^T])_4
\end{aligned}
$$

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